ISBN 82-553-0460-6
Mathematics
September 25

1981
No. 13

A NOTE ON SUFFICIENT AND NON-SUFFICIENT JETS

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PREPRINT SERIES - Matematisk institutt, Universitetet i Oslo

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Introduction

Let $E_{[k]}$ denote the set of germs of C^k mappings $f:(R^n,0) \to (R^p,0)$. Given a jet $z \in J^r(n,p)$, we say that $f \in E_{[r+s]}, s > 0$, is a realization of z if $j^r f(0) = z$. z is C^0-sufficient in $E_{[r+s]}$ if all realizations are C^0 equivalent. That is, if f and g are realizations of z in $E_{[r+s]}$, then there exists a germ of a homeomorphism $h:(R^n,0) \to (R^n,0)$ such that $f = g \circ h$. In [11] Thom conjectures that if z is not sufficient, then z admits an infinite number of realizations which are not topologically equivalent. When $p = 1$, and we consider C^0-sufficiency in $E_{[r]}$ and $E_{[r+1]}$, this is proved in [3]. On the other hand, Thom's conjecture becomes false when we consider sufficiency in $E_{[r+s]}$, $s > 1$. In [6], there is given an example of a $z \in J^6(2,1)$ for which all realizations in $E_{[8]}$ fall into two distinct C^0 equivalent classes.

In the case $p = 1$, C^0-sufficiency in $E_{[r]}$ (resp. $E_{[r+1]}$) is equivalent with v-sufficiency in $E_{[r]}$ (resp. $E_{[r+1]}$).(See [1].) Recall that a jet $z \in J^r(n,p)$ is v-sufficient in $E_{[r+s]}$ if the set germs $f^{-1}(0)$, $g^{-1}(0)$ are homeomorphic for any two C^{r+s} realizations f and g. Hence when $p = 1$, Thom's conjecture is proved by showing that if $z \in J^r(n,1)$ is not v-sufficient in $E_{[r]}$ (resp. $E_{[r+1]}$), then it admits an infinite number of realizations having non homeomorphic zero-sets.

In the case $p > 1$, a jet $z \in J^r(n,p)$ can be v-sufficient and still not C^0-sufficient. Hence the proof of Thom's conjecture of non-sufficiency in $E_{[r]}$ or $E_{[r+1]}$ does not carry automatically over from $p = 1$ to $p > 1$. In §1 of this paper we will give a proof of this conjecture also for $p > 1$. Here we follow an idea inspired by Wilson [16], to use Whitney Extension Theorem to construct certain realizations of a non-sufficient jet. Compare this with the proof in case $p = 1$ ([3]), which is of more analytic nature.

In [9], it is proven that certain characterizations of v-sufficiency is equivalent with some regularity conditions for stratifications. This gives a geometric explaination of the example in [6]. In §2 we will define analogue conditions, which will be relevant for C^0-sufficiency when $p > 1$, and prove corresponding results.

§1. Non sufficiency in $E_{[r]}$, $E_{[r+1]}$ when $p > 1$.

Let us first recall some results about sufficiency of jets which are proved in [2]. Let $z \in J^r(n,p)$, and consider z as a polynomial mapping $z = (z_1,\dots,z_p) : (R^n,0) \to (R^p,0)$ of degree r. Let $d(\text{Grad } z_i(x), \sum_{i\neq j} R \text{ Grad } z_j(x))$ denote the distance from $\text{Grad } z_i(x)$ to the linear subspace in R^n spanned by the $\text{Grad } z_j(x)$' s, $j \neq i$. Put $d(\text{Grad } z_1(x),\dots,\text{Grad } z_i(x)) = \min d(\text{Grad } z_i(x), \sum_{i\neq j} R \text{ Grad } z_j(x))$. Then the following theorem is proved in [2].

<u>Theorem (Bochnak, Kucharz [2]). Let $z \in J^r(n,p)$ be a jet with a critical point at 0.</u>

A <u>The following conditions are equivalent.</u>

i) <u>z is sufficient in</u> $E_{[r]}$

ii) $\exists C,\varepsilon > 0$ <u>such that</u>

$d(\text{Grad } z_1(x),\dots,\text{Grad } z_p(x)) > C\|x\|^{r-1}$ <u>for</u> $\|x\| < \varepsilon$.

iii) $\forall f \in E_{[r]}$ <u>with</u> $j^r f(0) = z$, 0 <u>is an isolated critical point of f.</u>

B The following conditions are equivalent.

i) z is sufficient in $E_{[r+1]}$.

ii) $\exists\, C,\delta,\varepsilon > 0$ such that

$$d(\text{Grad } z_1(x),\dots,\text{Grad } z_p(x)) \geq C\|x\|^{r-\delta} \quad \text{for} \quad \|x\| < \varepsilon.$$

Note that in [2] part A of the theorem above is announced for jets with $j^1z(0) = 0$. The proof, however, is valid for all z which have critical point at 0.

Now let us announce the main result of this section:

Theorem 1 Assume $z \in J^r(n,p)$ is not C^0-sufficient in $E_{[r]}$ (resp. $E_{[r+1]}$). Then there exists a sequence $\{f_k\}$ with $f_k \in E_{[r]}$ (resp. $f_k \in E_{[k+1]}$), and $j^rf_k(0) = z$, such that f_k and f_j are not C^0 equivalent when $k \neq j$.

Remark. When $n < p$, any jet $z \in J^r(n,p)$ is not sufficient. If $n \geq p$, and $z \in J^1(n,p)$, is not sufficient, then z is not surjective. In both these cases, it follows that im z has measure zero in R^p, and it is possible to construct a sequence $\{f_k\}$ of mappings realizing z, with im $f_k \neq$ im f_j, when $k \neq j$. This will show that Theorem 1 is true also in these cases. We will, however, omit the proof of this, and stick to the case $n \geq p$, and $r > 1$.

Let us first prove Theorem 1 in the case $E_{[r]}$. We will start by proving a lemma, which is a C^r version of Wilson's Lemma 3.3 in [16]. First identity $J^r(n,p)$ with a Euclidean space in an obvious way.

Lemma 1 Let $\{x_i\}$, $x_i \neq 0$, be a sequence in R^n converging to 0, and let $\{(y_i,z_i)\}$ be a sequence in $R^p \times J^r(n,p)$ such that $y_i = o(\|x\|^r)$, $z_i = o(\|x_i\|^{r-1})$. Then there exists a C^r-map $h : R^n \to R^p$ such that $j^rh(0) = 0$, and $(h(x_i),j^rh(x_i)) = (y_i,z_i)$ holds for a subsequence of $\{x_i\}$.

The proof of this lemma is almost a copy of the proof of Lemma 3.3 in [16]. Since this is not yet published, we will give the details.

Proof of Lemma 1. By passing to a subsequence if necessary, we may assume that for i,j, $j>i$ we have: $\|x_i\| < 2\|x_i - x_j\|$, $\|x_j\| < \|x_i\|$. Let $K = \{0\} \cup \bigcup_i \{x_i\}$. Then $\{(y_i, z_i)\}$ defines a Taylorfield on $\{x_i\}$, which we extend to K by adding the zero series at 0. Call this field $F = (F^k)_{|k|\leq r}$. We will prove that F is a C^r Whitney-field. Then the lemma follows from Whitney's Extension Theorem. (Here and throughout the article we will use the notation, and results in [12] concerning Whitney fields.)

Let $k = (k_1, \ldots, k_n)$, $|k| \leq r$, denote any multiindex. We have to prove that

$$(R_x^r F)^k(y) = F^k(y) - D^k \circ T_x^r F(y) = o(\|x-y\|^{r-|k|}) \quad \text{when} \quad x,y \in K.$$

Note that since $y_i = o(\|x_i\|^r)$, $z_i = o(\|x_i\|^{r-1})$ and $\|x\| \leq 2\|x-y\|$ if $x,y \in K$, we have that

$$F^0(x) = o(\|x-y\|^r) \quad \text{and} \quad F^k(x) = o(\|x-y\|^{r-1}) \quad \text{if} \quad |k|>0.$$

If follows that

$$(R_x F)^0(y) = F^0(y) - \sum_{|\ell|\leq r} \frac{F^\ell(x)}{\ell!}(y-x)^\ell = o(\|x-y\|^r.$$

When $|k|>0$ we have that

$$(R_x F)^k = F^k(y) - \sum_{|\ell|\leq r-|k|} \frac{F^{k+\ell}(x)}{\ell!}(y-x)^\ell = o(\|x-y\|^{r-1}).$$

This shows that F is a Whitney field, hence the lemma follows.

Now let us assume that $z \in J^r(n,p)$ is not sufficient. It follows that there exists a sequence $\{x_i\}$ tending to 0, such that $d(\text{Grad } z_1(x_i), \ldots, \text{Grad } z_p(x_i)) = o(\|x_i\|^{r-1})$. Let $\Sigma \subset J^1(n,p)$ be the set of singular jets. It is easy to see that $d(\text{Grad } z_1(x_i),\ldots,\text{Grad } z_p(x_i)) < d(j^1z(x_i),\Sigma)$ (the distance from $j^1z(x_i)$ to Σ). Consider the set $(\pi_1^2)^{-1}(\Sigma)$, where $\pi_1^2 : J^2(n,p) \to J^1(n,p)$ is the canonical projection. In the set $(\pi_1^2)^{-1}(\Sigma)$, the Boardmanstratum $\Sigma^{(n-p+1,0)}$ is of codimension 0, but all other Boardmanstrata have greater codimension. This follows from the formula of the codimension of the Boardmanstratum given in [10]. It follows that $\Sigma^{(n-p+1,0)}$ is open and dense in $(\pi_1^2)^{-1}\Sigma$. Let $\pi_2^r : J^r(n,p) \to J^2(n,p)$ be the canonical projection. It follows from above that the set $W = (\pi_2^r)^{-1}(\Sigma^{(n-p+1,0)})$ is open and dense in $(\pi_1^r)^{-1}(\Sigma)$. The jets in $(\pi_2^r)^{-1}(\Sigma^{(n-p+1,0)}$ are folds, which have a normal form given in [5] p. 88. From this follows that they are not C^0 equivalent with submersions.

Now, since $d(j^1z(x_i),\Sigma) = o(\|x_i\|^{r-1})$, it follows that we can find a sequenct $\{z_i\}$ in $J^r(n,p)$ such that $z_i = o(\|x_i\|^{r-1})$, and $j^rz(x_i)+z_i \in W$. By Sard's Theorem, find a sequence $\{y_i^1\}$ in R^p such that $y_i^1 = o(\|x_i\|^r)$, and $y_i^1+z(x_i)$ is a regular value for z. By Lemma 1, we can find a C^r mapping $h_1: (R^n,0) \to (R^p,0)$ such that $j^rh_1(0) = 0$, and $(h_1(x_i),j^rh(x_i)) = (y_i^1,z_i)$ holds on a subsequence of $\{x_i\}$, which we still denote by $\{x_i\}$. Put $f_1 = z+h_1$. Since $f_1(x_i) = y_i^1+z(x_i)$, f_1 is a fold around x_i, and $y_i^1+z(x_i)$ is a regular value for z, it follows that z and f_1 are not C^0 equivalent. To end the proof assume we have constructed realizations $f_1,\ldots,f_k$ of z which are not C^0 equivalent. By repeating the arguments above, find a sequence $\{y_i^{k+1}\}$ in R^p such that $y_i^{k+1} = o(\|x_i\|^r)$, and $z(x_i) + y_i^{k+1}$ is a regular value for $f_1,\ldots,f_k$. Then find a C^r mapping f_{k+1} such that $j^rf_{k+1}(0) = z$, and $(f_{k+1}(x_i),j^rf_{k+1}(x_i)) = (z(x_i)+y_i^{k+1},j^rz(x_i)+z_i)$, on a subsequence of $\{x_i\}$. It follows that f_{k+1} is not C^0 right equivalent with any f_i, $i < k$. In this way we can construct the sequence $\{f_k\}$, and prove Theorem 1 in the case $E_{[r]}$.

To prove Theorem 1 in the case $E_{[r+1]}$, it is enough to construct a realization f of z in $E_{[r+1]}$, such that 0 is not an isolated critical point. From the theorem of Bochnak, Kucharz follows that $j^{r+1}f(o)$ is not sufficient in $E_{[r+1]}$, and from above follows that we can construct an infinite number of not equivalent C^{r+1} realizations of $j^{r+1}f(o)$, which also are C^{r+1} realizations of z. To find such a f we need a lemma:

Lemma 2. Let $\{x_i\}$, $x_i \neq 0$, be a sequence of points in R^n converging to 0. Let $\alpha : \{x_i\} \to R$ be a function for each j, $1 \leq j \leq n$, such that $\alpha_j(x_i) = O(\|x_i\|^r)$. Then there exists a C^{r+1} function $f\colon R^n \to R$ such that $j^r f(o) = 0$, and $\frac{\partial f}{\partial x_j}(x_i) = \alpha_j(x_i)$ holds for a subsequence of $\{x_i\}$.

Proof. Let $x_i = (x_i^1,\ldots,x_i^n)$. By passing to a subsequence if necessary, we can assume that $\{\frac{\alpha^j(x_i)}{\|x_i\|^r}\}$, $\{\frac{x_i^j}{\|x_i\|}\}$ $j = 1,\ldots,n$ are convergent sequences.

Put $\alpha_j = \lim\limits_{i\to\infty} \frac{\alpha_j(x_i)}{\|x_i\|^r}$, $v_j = \lim\limits_{i\to\infty} \frac{x_i^j}{\|x_i\|}$, and $\alpha = (\alpha_1,\ldots,\alpha_n)$, $v = (v_1,\ldots,v_n)$. Since $v \neq 0$ assume that $v_1 \neq 0$.

Since $\{x_i\}$ is convergent to 0, it is not hard to see that it is possible to choose a subsequence of $\{x_i\}$ such that the following holds:

For each $n \in N$ we can find $N_n \in N$ such that if x_i, x_q are in the subsequence, and $q > i > N_n$, we have:

$$*)\quad \left|\frac{\|x_i\|}{\|x_i - x_q\|} - 1\right| < \frac{1}{n}\,, \quad \frac{\|x_q\|}{\|x_i - x_q\|} < \frac{1}{n}\,,$$

and either the j'th component of the subsequence are identically zero or

$$**)\quad \left|\frac{x_i^j}{x_i^j-x_q^j}-1\right| < \frac{1}{n}\ ,\quad \left|\frac{x_q^j}{x_i^j-x_q^j}\right| < \frac{1}{n}\ .$$

Let us still denote this subsequence by $\{x_i\}$. Put $K = \{0\} \cup \cup_i \{x_i\}$. We will define a Taylorfield on K and show that it is a Whitney-field.

To define the Taylorfield $F = (F^k)_{|k| \le r+1}$ on K, consider first the multiindexes

$$\begin{array}{l} k^1 = (r+1,0,\ldots,0) \\ \vdots \\ k^j = (r,0,\ldots,1,\ldots,0) \qquad\qquad (1 \text{ at } j\text{'th place}) \\ \vdots \\ k^n = (r,\ 0\ ,\ldots,0,1) \end{array}$$

Put

$$F^{k^1}(0) = \frac{r!\alpha_1}{(v_1)^r} - \sum_{j=2}^{p} \frac{rr!\alpha_j v_j}{(v_1)^{r+1}}$$

and

$$F^{k^j}(0) = \frac{r!\alpha_j}{(v_1)^r}\ ,\ \text{when}\quad j = 2,\ldots,n.$$

For all other multiindexes put $F^k(0) = 0$.

Define

$$P(x) = P^0(x) = \sum_{|k| \le r+1} \frac{F^k(0)}{k!}\, x^k$$

and

$$P^k(x) = D^k P(x) \quad \text{for each} \quad k,\ |k| \le r+1.$$

At last, if $k = (j) = (0,\ldots,1,\ldots,0)$ (1 at j'th place), put

$$F^k(x_i) = \alpha^j(x_i)$$

and put

$$F^k(x_i) = P^k(v)\|x_i\|^{r+1-|k|}$$

otherwise.

To prove that F is a Whitneyfield we have to show that

$$\frac{(R_x^{r+1}F)^k(y)}{\|x-y\|^{r+1-|k|}} = \frac{F^k(y)-D^k \circ T_x^{r+1}F(y)}{\|x-y\|^{r+1-|k|}} \to 0 ,$$

when $x,y \in K$ and $\|x-y\| \to 0$. This will follow from calculating some limits. These calculations will mostly be based on the inequalities *) and **) above.

Assume first that we consider points $x,y \in K$ with $x = x_i$, $y = x_q$ and $q > i$.

We have that

$$\frac{F^k(y)}{\|x-y\|^{r+1-|k|}} = \begin{cases} P^k(v)\left(\frac{\|x_q\|}{\|x_i-x_q\|}\right)^{r+1-|k|}, & \text{when } k \neq 1. \\ \frac{\alpha^j(x_q)}{\|x_i-x_q\|^r} = \frac{\alpha^j(x_q)}{\|x_q\|^r}\left(\frac{\|x_q\|}{\|x_i-x_q\|}\right)^r, & \text{when } k = (j). \end{cases}$$

Because $\frac{\|x_q\|}{\|x_i-x_q\|} \to 0$ when $i,q \to \infty$ and $\frac{\alpha_j(x_q)}{\|x_q\|^r}$ is bounded, we have that $\frac{F^k(y)}{\|x-y\|^{r+1-|k|}} \to 0$ when $|k| \neq r+1$ for such points x,y.

When $|k| = r+1$ we have that $\frac{F^k(y)}{\|x-y\|^{r+1-|k|}} = P^k(v)$. Since $P^k(0) = 0$ when $|k| < r+1$, and $P^k(0) = P^k(v)$ for $|k| = r+1$, we conclude that $\frac{F^k(y)}{\|x-y\|^{r+1-k}} \to P^k(0)$ when $\|x-y\| \to 0$.

On the other hand we have that

$$\frac{D^k \circ T_x^{r+1}F(y)}{\|x-y\|^{r+1-|k|}} = \frac{\sum\limits_{0\leq|\ell|\leq r+1-|k|} \frac{1}{\ell!}F^{\ell+k}(x_i)(x_q-x_i)^\ell}{\|x_i-x_q\|^{r+1-|k|}} =$$

$$= \sum_{0\leq|\ell|\leq r+1-|k|} \frac{F^{\ell+k}(x_i)}{\ell!\,\|x_i-x_q\|^{r+1-|\ell|-|k|}} \frac{(x_q-x_i)^\ell}{\|x_i-x_q\|^{|\ell|}} .$$

It is easy to see that $\frac{(x_q - x_i)^\ell}{\|x_i - x_q\|^{|\ell|}} \to (-v)^\ell$ when $i,q \to \infty$.

We also have that for $|\ell+k| \neq 1$, then

$$\frac{F^{\ell+k}(x_i)}{\|x_i - x_q\|^{r+1-|\ell|-|k|}} = \frac{P^{\ell+k}(v)\|x_i\|^{r+1-|\ell|-|k|}}{\|x_i - x_q\|^{r+1-|\ell|-|k|}} \to P^{\ell+k}(v) \quad \text{when}$$

$i,q \to \infty$. When $\ell+k = (j)$ we have that

$$\frac{F^{\ell+k}(x_i)}{\|x_i - x_q\|^{r+1-|\ell|-|k|}} = \frac{\alpha_j(x_i)\|x_i\|^r}{\|x_i\|^r\|x_i - x_q\|^r} \to \alpha_j$$

when $i,q \to \infty$. It is easily seen that $\frac{\partial P}{\partial x_j}(v) = \alpha_j$, hence when $\|x-y\| \to 0$ we have that

$$\frac{D^k \circ T_x^{r+1} F(y)}{\|x-y\|^{r+1-|k|}} \to \sum_{0<|\ell|<r+1-|k|} \frac{P^{\ell+k}(v)}{\ell!}(-v)^\ell = P^k(0)$$

because P is analytic.

Hence $\frac{(R_x^{r+1}F)(y)}{\|x-y\|^{r+1-k}} \to P^k(0) - P^k(0) = 0$ when $\|x-y\| \to 0$ and $x = x_i$, $y = x_q$ $q > i$.

If we interchange x and y, considerations similar to those above give that

$$\frac{F^k(x)}{\|y-x\|^{r+1-|k|}} \to P^k(v) \quad \text{and} \quad \frac{D^k \circ T_y^{r+1} F(x)}{\|y-x\|^{r+1-|k|}} \to P^k(v)$$

when $\|x-y\| \to 0$. Hence $\frac{(R_y^{r+1})^k(x)}{\|x-y\|^{r+1-|k|}} \to 0$ when $\|x-y\| \to 0$ in this case too. The case where x or y is 0 can be treated in a similar manner. This proves that F is a Whitneyfield, and the lemma follows from Whitney Extension Theorem.

Now let us end the proof of Theorem 1 in the case $E_{[r+1]}$. Put $h_i(x) = d(\mathrm{Grad}\, z_i(x), \sum_{i \neq j} \mathbb{R}\, \mathrm{Grad}\, z_j(x))$. Since z is not sufficient in $E_{[r+1]}$, we can assume that for each $\delta > 0$ there exists a sequence $\{x_i\}$ tending to 0 such that $h_1(x_i) = o(\|x_i\|^{r-\delta})$. Note that by [7] p.118, $(h_1)^2$ is a bounded rational function. It follows from the Tarski-Seidenberg Theorem that the set

$V = \{(u,v) \in \mathbb{R}^2 | (u,v) = ((h_1)^2(x), \|x\|^2), x \in \mathbb{R}^n\}$ is semialgebraic. It is not hard to see that the set $\{(u,v) \in V | u = \min_{\|x\|^2=v} (h_1)^2(x)\} - \{0\}$ is a component of $(V-V^o)-\{0\}$, hence semialgebraic. It follows from the Curve Selection Lemma that there exists an analytic arc $\beta:[0,\varepsilon] \to V$ such that $\beta(0) = 0$ and $\beta(t) \in \{x | h_1(x) = \min_{\|w\|=\|x\|} h_1(w)\}$. Assume that $|\beta(t)| \sim t^q$ and that $|h_1(\beta(t)| \sim t^s$. (Note that from the expression of h_1 given in [7], it will follow that s is an integer.) From the theorem of Bochnak, Kucharz follows that $s/q > r$. Let $\{x_i\}$ be a sequence on $\beta([0,\varepsilon])$ converging to 0. Then we must have $h_1(x_i) = O(\|x_i\|^r)$. Now since $h_1(x_i) > d(j^1z(x_i),\Sigma)$, it follows that we can find a sequence $\{z_i\}$ in $J^1(n,p)$ such that $z_i = O(\|x_i\|^r)$ and $j^1z(x_i)+z_i \in \Sigma$.

Now apply Lemma 2 for the p components of z_i, to find a C^{r+1} map $h: \mathbb{R}^n \to \mathbb{R}^p$ such that $j^rh(0) = 0$, and $j^1h(x_i) = z_i$ holds on a subsequence of $\{x_i\}$.

Put $f = z+h$. Then f is the desired realization of z with singular points on a subsequence of $\{x_i\}$. This completes the proof of Theorem 1 in the case $E_{[r+1]}$.

Remark. From the arguments above follows directly that sufficiency of z in $E_{[r+1]}$ is equivalent with the condition that every C^{r+1} realization of x admits 0 as an isolated critical point.

§ 2. Geometric conditions of sufficiency.

As in [9], consider $z \in J^r(n,p)$ as a polynomial map $z = (z_1,\dots,z_p):(\mathbb{R}^n,0) \to (\mathbb{R}^p,0)$ of degree r, and define

$$F(x,\lambda) = (F_1(x,\lambda^{(1)}),\dots,F_p(x,\lambda^{(p)}))$$

where

$$F_i(x,\lambda^{(i)}) = z_i(x)+ \sum_{|\alpha|=r} \lambda_\alpha^{(i)} x^\alpha, \quad 1 \leq i \leq p.$$

Consider the Euclidean space $\mathbb{R}^n \times \Lambda$ where Λ is formed by the $\lambda_\alpha^{(i)}$'s. As explained in [9], the Grad F_i's, $1 \leq i \leq p$, are linearly independent at points $(x,\lambda) \in \mathbb{R}^n \times \Lambda$ where $x \neq 0$. It follows that $F^{-1}(F(x,\lambda))$ is a manifold of codimension p for $x \neq 0$. Now consider the following conditions:

(w_F). There exists a neighbourhood U of 0 in $\mathbb{R}^n \times \Lambda$ and $C > 0$ such that for $(x,\lambda) \in U$, $x \neq 0$, we have

$$d(0\times\Lambda, T_{(x,\lambda)}F^{-1}(F(x,\lambda))) \leq C\|x\|.$$

(Recall that when V,W are linear subspaces of $\mathbb{R}^n$ then $d(V,W) = \sup_{\substack{v\in V\\ \|v\|=1}} \inf_{w\in W} \|v-w\|$.)

(t_F^s). Let M_s^n denote any C^s submanifold, $s \geq 1$, of $\mathbb{R}^n \times \Lambda$ of dimension n with $0 \in M_s^n$. Assume that M_s^n is transverse to $0 \times \Lambda$ at 0, then there exists a neighbourhood U of 0 in $\mathbb{R}^n \times \Lambda$ such that when $(x,\lambda) \in U \cap M_s^n$, $x \neq 0$, then M_s^n is transverse to $F^{-1}(F(x,\lambda))$ at (x,λ).

Note that the conditions (w_F) and (t_F^s) are generalizations of Verdiers Condition and the Trotman Condition (t^s), (see [9]) where we also compare $0 \times \Lambda$ with the manifolds $F^{-1}(a)$, $a \neq 0$, in a neighbourhood of 0.

Now we have:

<u>Theorem 2.</u> <u>Let</u> $z \in J^r(n,p)$ <u>be an</u> r <u>jet.</u> <u>Assume</u> $s \in \mathbb{N}$, $s \geq 1$. <u>Then the following conditions below</u> I <u>and</u> II <u>are equivalent respectively.</u>

I

(i) <u>z is</u> C^0 <u>sufficient in</u> $E_{[r]}$.

(ii) <u>The condition</u> (w_F) <u>is satisfied</u>.

II

(i) <u>The condition</u> (t_F^s) <u>is satisfied</u>.

(ii) <u>Any</u> $w \in J^{r+s}(n,p)$ <u>with</u> $j^r w(0) = z$ <u>is</u> C^0 <u>sufficient in</u> $E_{[r+s]}$.

(iii) <u>z admits only a finite number of</u> C^{r+s} <u>realizations which are not</u> C^0 <u>equivalent</u>.

(iv) <u>Any</u> C^{r+s} <u>relazation</u> f <u>of z admits</u> 0 <u>as an isolated critical point</u>.

(v) <u>For any family of</u> C^s <u>functions</u> $\lambda_\alpha^{(i)}(x)$, $|\alpha| = r$, $1<i<p$, $\lambda_\alpha^{(i)}(0) = 0$, <u>the</u> C^s <u>mapping</u> $F(x,\lambda(x))$ <u>admits</u> 0 <u>as an isolated critical point</u>.

<u>Remark</u>. Inspired by Theorem A in [9], the author was a while tempted to guess that sufficiency in $E_{[r+1]}$ was equivalent with the condition (a_F) below, which is a generalization of the Whitney (a) condition.

(a_F). Assume $\{(x_i,\lambda_i)\}$ is a sequence with $x_i \neq 0$ tending to 0 in $\mathbb{R}^n \times \Lambda$. Assume that $T_{(x_i,\lambda_i)}F^{-1}(F(x_i,\lambda_i)) \to \tau$ in the appropriate Grassmanian, then $\tau \supseteq 0\times\Lambda$.

The equivalence between sufficiency in $E_{[r+1]}$ and the condition (a_F) is however false. A counterexample is the following: Consider $z \in J^4(2,1)$, $z = x_1^3-3x_1x_2^3$. From calculations in [8] p. 228 it follows that z is sufficient in $E_{[5]}$ but not in $E_{[4]}$. An easy calculation will show that (a_F) breaks down along the curve $x_1 = t^3$, $x_2 = t^2$, $\lambda_\alpha = 0$, for $\alpha \neq (0,4)$, and $\lambda_{(0,4)} = \frac{9}{4}t$. From this example it is also easy to construct counterexamples when $p > 1$. It is however possible to prove that (a_F) implies sufficiency in $E_{[r+1]}$. We will here omit the details.

Let us now prove part I of Theorem 2. Let $N_i(x,\lambda) = \mathrm{Grad}F_i(x,\lambda) - P_i(x,\lambda)$, where $P_i(x,\lambda)$ is the projection of $\mathrm{Grad}F_i(x,\lambda)$ onto the linear space spanned by the $\mathrm{Grad}F_j(x,\lambda)$'s $j \neq i$. Then, using formula (3.3) of [7], the distance from the unit vector $\frac{\partial}{\partial\lambda_\alpha^{(i)}}$ to the tangentspace $T_{(x,\lambda)}F^{-1}(F(x,\lambda))$ when $x \neq 0$ is

$$\delta_\alpha^{(i)}(x,\lambda) = \left\| \sum_{j=1}^{p} \frac{\partial}{\partial\lambda_\alpha^{(i)}} \circ \mathrm{Grad}\ F_j(x,\lambda) \frac{N_j(x,\lambda)}{\|N_j(x,\lambda)\|^2} \right\| = \frac{|x^\alpha|}{\|N_j(x,\lambda)\|}$$

To prove (i) $\Rightarrow$ (ii) assume z is sufficient in $E_{[r]}$. From the theorem of Bochnak, Kucharz follows that $d(\mathrm{Grad}z_1(x),\ldots,\mathrm{Grad}z_p(x)) > C\|x\|^{r-1}$ for some $C > 0$ when $\|x\|$ is small. As in the proof of Lemma 4.3 [7], it follows that $d(\mathrm{Grad}F_1(x,\lambda),\ldots,\mathrm{Grad}F_p(x,\lambda)) > \frac{C}{2}\|x\|^{r-1}$ in a sufficiently small neighbourhood of 0. Since $\|N_i\| > d(\mathrm{Grad}F_1,\ldots,\mathrm{Grad}F_p)$, it follows that $\delta_\alpha^{(i)}(x,\lambda) < \frac{2}{C}\|x\|$. Since $0\times\Lambda$ is spanned by the orthnormal vectors $\frac{\partial}{\partial\lambda_\alpha^{(i)}}$, the condition (w_F) is satisfied.

To prove (ii) $\Rightarrow$ (i) assume that x is not sufficient in $E_{[r]}$. Then there exists a C^r function $h\colon \mathbb{R}^n \to \mathbb{R}^p$, with $j^r h(0)=0$ such that $f = z+h$ have a sequence $\{x_i\}$, of critical points tending to 0. Hence we can assume that on this sequence we have $\mathrm{Grad}\ f_1 = \sum_{j=2}^{p} \beta_j\ \mathrm{Grad}\ f_j$ where the β_j's are numbers with $|\beta_j| < 1$. Write $\mathrm{Grad}\ F_j = (\mathrm{Grad}_x F_j, \mathrm{Grad}_\lambda F_j)$. It follows from a short calculation that

$$\mathrm{Grad}_x F_1 = \sum_{j=2}^{p} \beta_j \, \mathrm{Grad}_x F_j + \sum_{j=2}^{p} \beta_j \, \mathrm{Grad}\, h_j - \mathrm{Grad}\, h_1$$

on the sequence $\{(x_i,0)\}$ in $\mathbb{R}^n \times \Lambda$. From this we get that:

$$\begin{aligned}\mathrm{Grad}\, F_1 &= (\mathrm{Grad}_x F_1, \mathrm{Grad}_\lambda F_1) \\ &= \sum_{j=2}^{p} \beta_j \, \mathrm{Grad}\, F_j + \sum_{j=2}^{p} \beta_j \, \mathrm{Grad}\, h_j - \mathrm{Grad}\, h_1 \\ &\quad + \mathrm{Grad}_\lambda F_1 - \sum_{j=2}^{p} \beta_j \, \mathrm{Grad}_\lambda F_j\end{aligned}$$

on the sequence $\{(x_i,0)\}$. From this follows that:

$$\begin{aligned}\|N_1\| &= \|\mathrm{Grad}\, F_1 - P_1\| \\ &< \Big\|\sum_{j=2}^{p} \beta_j \, \mathrm{Grad}\, h_j - \mathrm{Grad}\, h_1 + \mathrm{Grad}_\lambda F_1 - \sum_{j=2}^{p} \beta_j \, \mathrm{Grad}_\lambda F_j\Big\|.\end{aligned}$$

Now since $\|\mathrm{Grad}_\lambda F_j\| = O(\|x\|^r)$, $\|\mathrm{Grad}\, h_j\| = o(\|x\|^{r-1})$, $|\beta_j| < 1$, $1 < j < p$, it follows that $\|N_1\| = o(\|x\|^{r-1})$ on the sequence $\|(x_i,0)\|$. From this it is clear that for some α, $|\alpha| = r$, $\dfrac{|x^\alpha|}{\|N_1\|\|x\|}$ is not bounded on $\{(x_i,0)\}$. Since $\delta_\alpha^{(1)}(x_i,0) = \dfrac{|x_i^\alpha|}{\|N_1(x_i)\|}$, is the distance from $\dfrac{\partial}{\partial\lambda_\alpha^{(1)}}$ to $T_{(x_i,0)}F^{-1}(F(x_i,0))$ it follows that (w_F) fails along $\{(x_i,0)\}$, proving (ii) $\Rightarrow$ (i). Hence the proof of Theorem 2 part I is complete.

Part II of Theorem 2 is very similar to Theorem C in [9], and the proof is also very similar. We will only sketch it, pointing out the main differences from the proof of Theorem C. The proof of (i) $\Rightarrow$ (v) is almost a copy of (C.1) $\Rightarrow$ (C.5) in [9]. This is also the case for (i) $\Rightarrow$ (ii) which is similar to (C.1) $\Rightarrow$ (C.2). Note however that it is not necessary to have critical points along a Lojasiewicz arc, but only along a seequence tending ot 0. To prove (ii) $\Rightarrow$ (iii), note that (ii) implies that every $w \in J^{r+s}(n,p)$ with $j^r w(0) = z$ admits 0 as an isolated critical point. In the terminology of [2]

p. 118 this means that $w \in J_{\Sigma}^{r+s}(n,p)$. From Theorem 4 of [2] follows that there exists a partition of $J_{\Sigma}^{r+s}(n,p)$ in finitely many connected analytic varieties such that the jets occuring in the same variety are C^0 equivalent. It follows that $J_{\Sigma}^{r+s}(n,p)$ consists of finitely many C^0 equivalence classes. This will imply (iii). (Compare this with the proof of (C.2) $\Rightarrow$ (C.3) using Fukuda's Theorem. When $p > 1$ Fukuda's Theorem is not valid.)

(iii) $\Rightarrow$ (iv) is similar to (C.3) $\Rightarrow$ (C.4) using Theorem 1 in this article instead of the results in [3].

At last the proof of (iv) $\Rightarrow$ (v) is similar to (C.4) $\Rightarrow$ (C.5). The only obstacle is that we lack a theorem corresponding to Theorem A in [9]. (See the remark above.) From the remark below the proof of Theorem 1 in the case $E_{[r+1]}$ follows however, that it is sufficient to prove that sufficiency in $E_{[r+1]}$ implies the condition (v) when $s = 1$. To prove this, assume (v) fails for $z \in J^r(n,p)$. Then there exists a family of C^1 functions $\lambda_\alpha^{(j)}(x)$, $|\alpha| = r$, $1 \leq j \leq p$, and a sequence $\{x_i\}$ in $\mathbb{R}^n$ tending to 0, such that $f(x) = F(x,\lambda(x))$ has critical points along $\{x_i\}$. Hence we can assume that for each i there exists numbers β_j, $2 \leq j \leq p$ with $|\beta_j| \leq 1$ such that

$$\operatorname{Grad} f_1(x_i) = \sum_{j=2}^{p} \beta_j \operatorname{Grad} f_j(x_i)$$

where the f_j's are the component functions of f. From this we get:

$$\operatorname{Grad}_x F_1 = \sum_{j=2}^{p} \beta_j \operatorname{Grad}_x F_j + \sum_{j=2}^{p} \beta_j \sum_{\alpha} \frac{\partial F_j}{\partial \lambda_\alpha^{(j)}} \operatorname{Grad} \lambda_\alpha^{(j)}$$

$$- \sum_{\alpha} \frac{\partial F_1}{\partial x_\alpha^{(1)}} \operatorname{Grad} \lambda_\alpha^{(1)} \quad \text{along } \{(x_i,\lambda(x_i))\}.$$

Note that since the $\lambda_\alpha^{(j)}$'s are C^1 and $\lambda(0) = 0$, we have $\lambda_\alpha^{(j)}(x) = O(\|x\|)$. From this follows $\operatorname{Grad}_x F_j = \operatorname{Grad} z_j + O(\|x\|)^r$ along $(x_i,\lambda(x_i))$. Substituting this in the equality above, and using that

$\beta_j < 1$, $\|\mathrm{Grad}\lambda_\alpha^{(j)}\|$ is bounded and that $\frac{\partial F_j}{\partial \lambda_\alpha^{(j)}}(x,\lambda) = O(\|x\|^r)$ we get that

$$\mathrm{Grad}\ z_1 = \sum_{j=2}^{p} \beta_j\ \mathrm{Grad}\ z_j + O(\|x\|^r)$$

From this follows that $d(\mathrm{Grad}z_1,\ldots,\mathrm{Grad}z_p) = O(\|x\|^r)$ along $\{x_i\}$. It follows from the theorm of Bochnak, Kucharz that x is not sufficient in $E_{[r+1]}$. This completes the proof of (iv) $\Rightarrow$ (v) and Theorem 2.

Acknowledgement. The author would like to thank Per Holm, Andrew du Plessis and Les Wilson for helpful comments and communications.

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